It takes about 42 minutes!

For a detailed mathematical explanation read what follows:

§ 13. Newton’s law of gravitation
§ 14. Attraction due to a uniform circular ring at a point on its axis
§ 15. Attraction due to a uniform spherical shell
§ 16. Attraction due to a distribution of matter with spherical symmetry
Example
Continuation – with final mathematical result.

sense, and in the case of elliptic motion about a focus refers to the length of the semi-major axis. Thus K. 3 reads

$$\frac{\tau_1^2}{\tau_2^2}=\frac{a_1^3}{a_2^3},$$

where τ_1, τ_2 are the periods of revolution of two planets about the sun and a_1, a_2 are the lengths of the semi-major axes of their paths.

Now, with the usual notation, the area of an ellipse is πab, and $\frac{1}{2}h$ is the areal velocity with which the radius vector sweeps out the area of the ellipse. Consequently, the period τ is given by $2\pi ab/h$. Thus

$$\frac{\tau_1^2}{\tau_2^2}=\frac{a_1^2b_1^2}{h_1^2}\bigg/\frac{a_2^2b_2^2}{h_2^2}.$$

Since $b^2=a^2(1-e^2)=al$, it follows from the last two displayed equations that

$$\frac{h_2^2}{h_1^2}=\frac{b_2^2}{a_2}\bigg/\frac{b_1^2}{a_1}=\frac{l_2}{l_1}.$$

Hence,

$$\frac{h_2^2}{l_2}=\frac{h_1^2}{l_1}.$$

In other words, according to K. 3, the factor μ is the same for all planets in the solar system. We shall see later, however, that this statement is only approximately true.

Since the force between the sun and a planet is proportional to the mass of the planet, it seems natural to suppose that it is also proportional to the mass M of the sun. We conclude, therefore, that this force may be expressed in the form

$$\gamma\frac{Mm}{r^2},$$

where γ is a constant factor depending upon the units of mass and length employed, M is the mass of the sun, m is the mass of the planet and r is the distance between the sun and the planet.

§ 13. **Newton's law of gravitation.** With the aid of the calculus a lesser man than Newton might have proceeded thus far, but it required the genius and insight of Newton to see that the formula just obtained applies to any two particles in the universe, and to recognise that the force which holds the planets in their orbits is of the same character as that which makes the apple fall to the ground. Newton's law of gravitation asserts that *any two particles in the universe attract one another with a force*

$$\gamma\frac{m_1m_2}{r^2},$$

where m_1, m_2 *are the masses of the particles and where* r *is the distance between them.* γ is known as the **gravitational constant** and has the values

$6{\cdot}66\times10^{-8}$ in cm. gr. sec. system of units ;
$1{\cdot}05\times10^{-9}$ in ft. lb. sec. system of units.

So far we have followed the historical and physical line of approach. That is to say, we have made the universal law of gravitation seem plausible by appeals to astronomical observations of the solar system. We prefer, however, to take a different viewpoint and simply state the law of gravitation as an axiom. It will appear that the principal facts which can be deduced from this axiom do not conflict with observational evidence.

§ 14. **Attraction due to a uniform circular ring at a point on its axis.** Let the ring be of line density λ, that is, of mass λ per unit length, and let its radius be c. We shall calculate the gravitational attraction on a particle of unit mass situated on the axis of the ring and at distance p from its centre (fig. 5).

The attraction due to the mass element λdl is

$$\frac{\gamma\lambda dl}{p^2+c^2}.$$

The component of this force along the axis is

$$\frac{\gamma\lambda p dl}{(p^2+c^2)^{3/2}}.$$

When we integrate round the ring it is evident that, in view of symmetry, the resultant of the components perpendicular

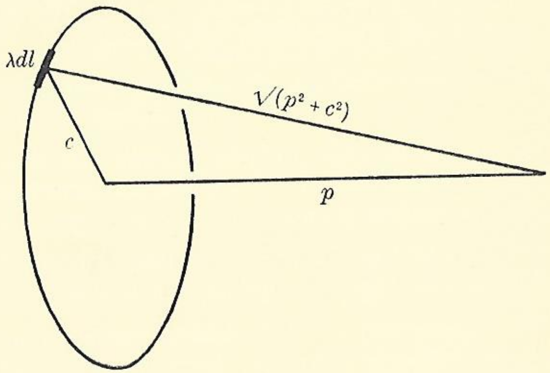

FIG. 5

to the axis is zero, while the resultant force along the axis is

$$\frac{2\pi c\gamma\lambda p}{(p^2+c^2)^{3/2}},$$

or

$$\frac{\gamma M p}{(p^2+c^2)^{3/2}},$$

where M is the total mass of the ring.

§ 15. Attraction due to a uniform spherical shell. Suppose that the shell be of radius a and of surface density σ, that is, of mass σ per unit area. We shall calculate the attraction on a particle of unit mass which is at distance r from the centre. To do this, we divide the shell up into elementary rings of line density $a\sigma d\theta$ and of radius $a \sin\theta$,

as shown in fig. 6. The centre of such a ring is at distance $r - a\cos\theta$ from the particle.

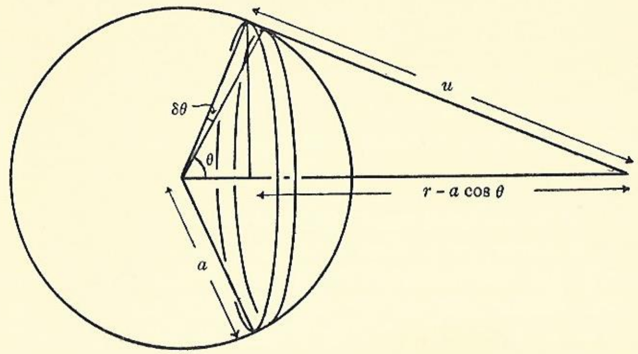

FIG. 6

The attraction due to this elementary ring is therefore, according to the result of § 14,

$$\frac{2\pi a \sin\theta\,(r - a\cos\theta)\gamma\sigma a d\theta}{\{(r-a\cos\theta)^2+(a\sin\theta)^2\}^{3/2}}.$$

Now, if we write

$$u^2 = (r-a\cos\theta)^2 + (a\sin\theta)^2 = r^2 - 2ar\cos\theta + a^2,$$

then

$$u du = ar\sin\theta\, d\theta$$

and

$$r - a\cos\theta = \frac{u^2+r^2-a^2}{2r}.$$

The attraction due to the elementary ring is therefore

$$\frac{\pi a\gamma\sigma}{r^2}\left(1+\frac{r^2-a^2}{u^2}\right)du.$$

Now if $r>a$, u varies from $r-a$ to $r+a$ and the attraction due to the whole shell is

$$\frac{\pi a\gamma\sigma}{r^2}\int_{r-a}^{r+a}\left(1+\frac{r^2-a^2}{u^2}\right)du=\frac{\pi a\gamma\sigma}{r^2}\left[u-\frac{(r^2-a^2)}{u}\right]_{r-a}^{r+a}$$
$$=\frac{4\pi a^2\gamma\sigma}{r^2}$$
$$=\frac{\gamma M}{r^2},$$

where M is the total mass of the shell.

If, however, $r<a$, then u varies from $a-r$ to $a+r$ and in this case the resultant force due to the shell vanishes, as the reader can easily verify. Thus the force on a particle of unit mass due to a uniform spherical shell of mass M is $\gamma M/r^2$ or 0, according as the particle is outside or inside the shell.

§ 16. Attraction due to a distribution of matter with spherical symmetry. Using the results of the previous section, we see that a particle of unit mass outside the distribution experiences a force $\gamma M/r^2$, where M is the total mass of the distribution and r is the distance of the particle from the centre of the distribution. In such a case the particle is acted upon by the same force as it would be if the whole mass of the distribution were concentrated at its centre. This means that the formula $\gamma m_1 m_2/r^2$ applies not only to two particles but also to any two spherical distributions of matter whose boundaries do not intersect, provided that r is understood to signify the distance between the centres of the distributions.

On the other hand, if the particle of unit mass is within the spherical distribution it will experience a force $\gamma M'/r^2$, where M' is the total mass contained within the sphere of radius r, for, as we have seen in the previous section, any elementary shell which includes the particle exerts no force on it. If the distribution be a uniform sphere of radius a,

then $M'=(r^3/a^3)M$ and the force exerted on the particle of unit mass at distance $r(<a)$ from its centre is

$$\frac{\gamma Mr}{a^3}. \qquad (16.1)$$

§ 17. Weight. From the foregoing arguments we see that, if the earth be assumed to have spherical symmetry, its gravitational attraction on a particle of mass m outside the earth is

$$\frac{\gamma Mm}{r^2},$$

where M is the mass of the earth and where r is the distance of the particle from the earth's centre. This force is called the **weight** of the particle and is written

$$mg,$$

where

$$g=\frac{\gamma M}{r^2}.$$

Now in any locality where an experiment is being performed the value of r is approximately constant and so in such cases g may be regarded as a constant. For different places on the earth's surface or at greatly differing altitudes, r may vary slightly and the value of g varies correspondingly. Its value at sea level for different places is given by the following table:

	c.g.s. units	f.p.s. units
Equator .	978·10	32·08
New York .	980·22	32·16
London .	981·19	32·19
North Pole .	983·21	32·25

Since the weight of a mass m is mg, it follows that the acceleration of this mass when falling freely is g. At a given place this acceleration is the same for all bodies irrespective

Example. As another illustration of simple harmonic motion we shall prove the remarkable but useless fact that if a straight smooth tunnel were bored through the earth, a particle released from rest at the earth's surface would slide through it in about 42 minutes, assuming that the density of the earth is uniform.

In fig. 10 let AB be the tunnel and let O be its mid-point. According to (16.1), when the particle is at P it is subject to a gravitational attraction $\gamma Mmr/a^3$ towards the centre of the earth. The component of the resulting acceleration in the direction OX is $-\gamma Mx/a^3$, where $x=OP$ and a is the radius of the earth. The equation of motion is therefore of the form

$$\ddot{x} = -n^2x,$$

where

$$n^2 = \frac{\gamma M}{a^3} = \frac{g}{a}.$$

The period of the motion is therefore $2\pi\sqrt{(a/g)}$ and the time for the particle to slide from one end of the tunnel to the other

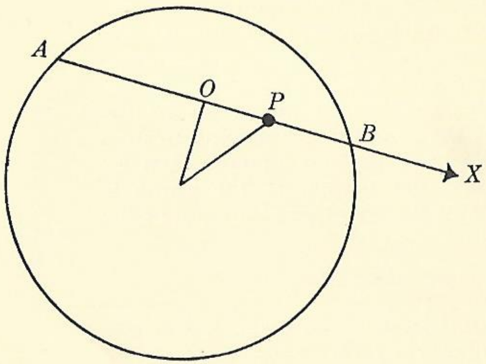

FIG. 10

is half of this value. Taking $g=981$ cms./sec.^{2} and $a=6370$ kms. as approximate values, we have

$$\tfrac{1}{2}\tau = \pi\sqrt{\frac{637000000}{981}} \text{ secs.,}$$

which is just over 42 minutes.

§ 32. Damped oscillator with a periodic applied force. The equation

$$\ddot{x} + k\dot{x} + n^2x = f\cos pt, \quad . \quad . \quad . \quad (32.1)$$

which we now discuss, is of interest not only in mechanics but also in the theory of alternating electric currents. In

the mechanical case it arises when a particle of mass m constrained to move along the x-axis is subject to three forces: (i) an elastic force $-mn^2x$, (ii) a resistance $-mk\dot{x}$ proportional to the velocity, (iii) a periodic applied force $mf\cos pt$ of period $2\pi/p$. When we have found the general solution of the equation (32.1) we may consider the special case in which the resistance is absent ($k=0$), or the case in which the applied force is absent ($f=0$).

The solution of the equation (32.1) is, assuming $k<2n$,

$$x = ae^{-\frac{1}{2}kt}\cos(qt+\epsilon) + (f/R)\cos(pt-\alpha), \quad . \quad (32.2)$$

where a and ϵ are constants of integration, $q^2=n^2-\frac{1}{4}k^2$ and R and α are given by the right-angled triangle of fig. 11.

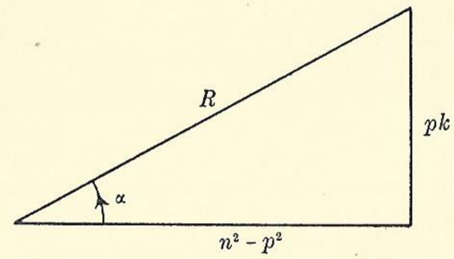

FIG. 11

This solution may be obtained by the standard method of dealing with such differential equations,* but the reader unfamiliar with this theory may satisfy himself by differentiation that the solution does, in fact, satisfy (32.1) and observe that since two constants a, ϵ of integration occur, the solution (32.2) is the most general one.

In the solution (32.2) the first term $ae^{-\frac{1}{2}kt}\cos(qt+\epsilon)$ is referred to as the **free** term since it occurs whether or not the applied force is present. The second term $(f/R)\cos(pt-\alpha)$ is called the **forced** term and it evidently has the same period $2\pi/p$ as the applied force, although it

* Ince, *Integration of Ordinary Differential Equations*, p. 97.